§ Change of Variables Thm.

$\left(\frac{\partial y_i}{\partial x_j}\right)_{i,j}$

$\|$

☆ Thm: Suppose $g: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a smooth bijective map and $\det Dg \neq 0$

$A \rightsquigarrow B$

$(x_1, \cdots, x_n) \rightsquigarrow (y_1, \cdots, y_n)$

"diffeomorphism", $A \cong B$.

Then

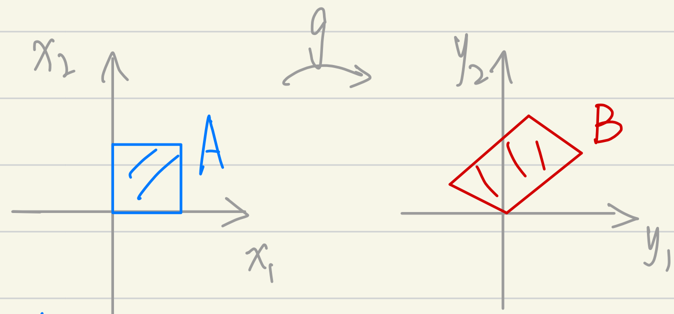
$$\boxed{\int_B f \, dV = \int_A f \circ g \cdot |\det Dg| \, dV}$$

$\|$

$$\int_B f(y_1, \cdots, y_n)\, dy_1 \cdots dy_n$$

$\|$

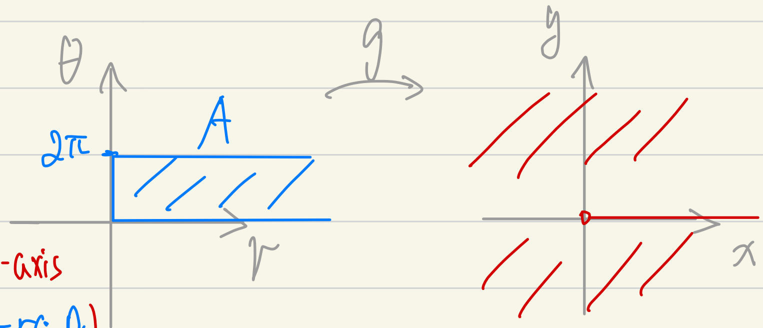
$$\int_A f \circ g(x_1, \cdots, x_n)\, |\det Dg|\, dx_1 \cdots dx_n$$



Ex 1: In dim = 1 Substitution of Variable:

$$\int_c^d f(y)\,dy \overset{y=g(x)}{=} \int_a^b f(g(x))\cdot g'(x)\,dx.$$

Ex 2: Polar Coordinates



$$g: A=(0,\infty)\times(0,2\pi) \longrightarrow \mathbb{R}^2 - \text{positive } x\text{-axis}$$

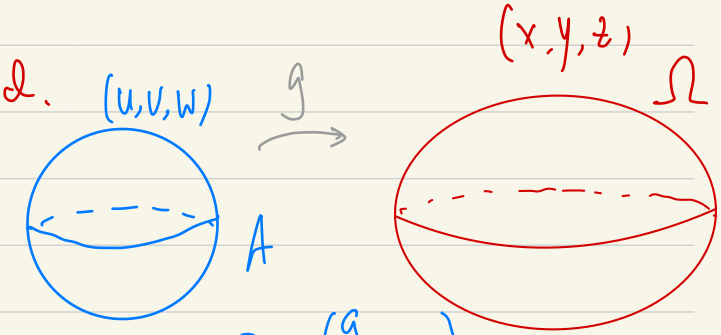
$$(r,\theta) \rightsquigarrow (x=r\cos\theta,\ y=r\sin\theta)$$

$$\det Dg = \det\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix} = r>0. \quad \Rightarrow \int_B f(x,y)\,dx\,dy = \int_A f(r,\theta)\,\underset{\substack{\| \\ |\det Dg|}}{r}\,dr\,d\theta$$

Ex 3: $\Omega: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$ ellipsoid.

Let $u=\frac{x}{a}$, $v=\frac{y}{b}$, $w=\frac{z}{c}$

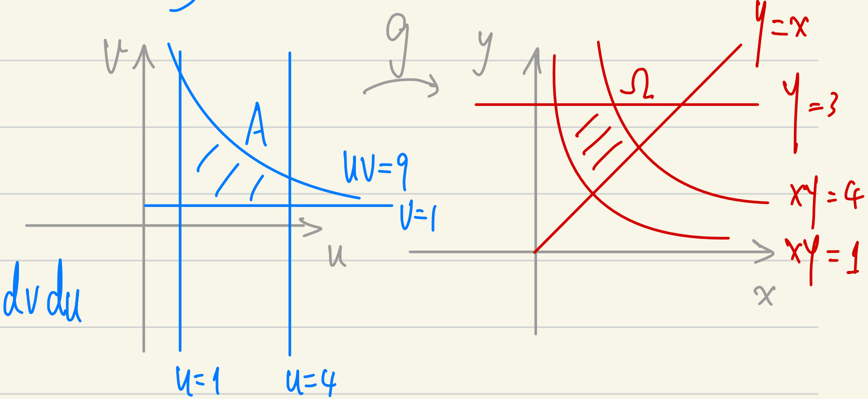
$A: u^2+v^2+w^2=1$ Sphere



$$\Rightarrow vol(\Omega)=\int_\Omega dV = \int_A |\det Dg|\, dV = abc\cdot\frac{4\pi}{3}$$

$$Dg=\begin{pmatrix} a & & \\ & b & \\ & & c\end{pmatrix}$$

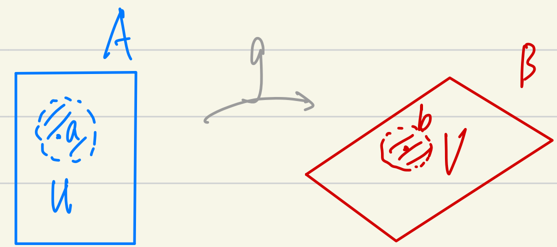
Ex 4: Let $\begin{cases} u=xy \\ v=\frac{y}{x}\end{cases}$ so $\begin{cases} x=\sqrt{\frac{u}{v}} \\ y=\sqrt{uv}\end{cases}$



$$\int_\Omega y\,dV = \int_A \sqrt{uv}\cdot \underbrace{|\det Dg|}_{=\frac{1}{2v}}\cdot dV = \int_1^4\int_1^{\frac{9}{u}} \frac{1}{2}\sqrt{\frac{u}{v}}\,dv\,du = \frac{13}{3}$$

Sketch of proof:

- $f = 1$ $\xrightarrow{\text{upper/lower sum}}$ Arbitrary f
- Local $\xrightarrow{\text{Partition of Unity}}$ Global

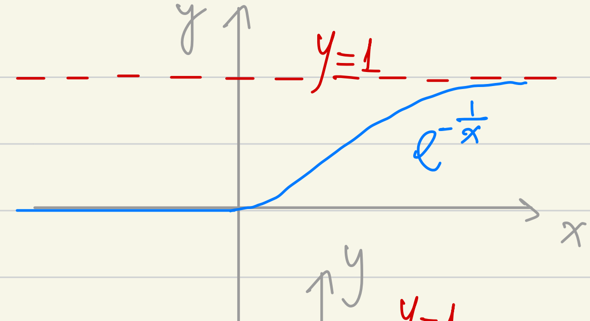


- Special g $\xrightarrow{\text{Composition + lin. transf.}}$ Arbitrary g
- $\dim = 1$ $\xrightarrow{\text{induction + Fubini}}$ $\dim = n$

Chain rule + Fund. Thm of Calculus

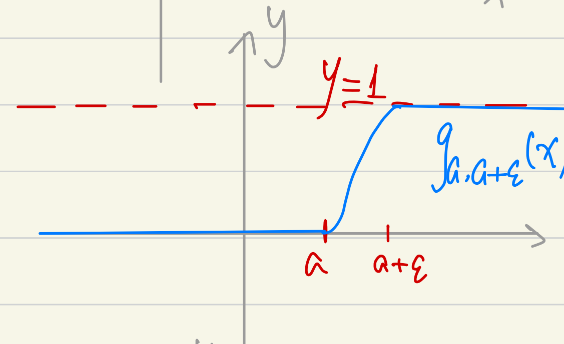
§ Partition of Unity

- $f_1(x) := \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases} \in C^{\infty}(\mathbb{R})$.



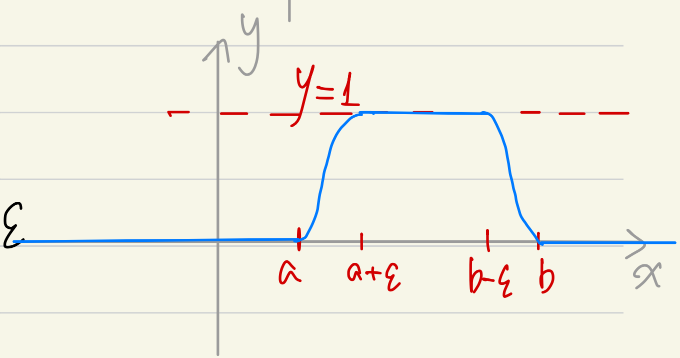
- $g_{a,a+\varepsilon}(x) := \dfrac{f(x-a)}{f(x-a)+f(a+\varepsilon-x)} = \begin{cases} 0 & x \leq a \\ 1 & x \geq a+\varepsilon \end{cases}$

$(\varepsilon > 0)$



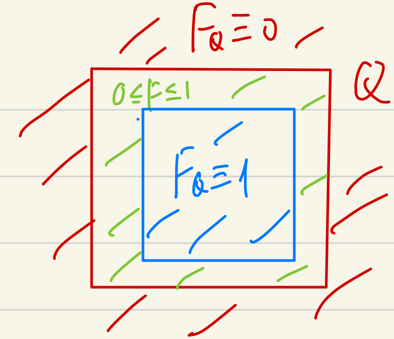
- $h_{a,b}(x) := g_{a,a+\varepsilon}(x) \cdot (1 - g_{b-\varepsilon,b}(x))$

$(a < a+\varepsilon < b-\varepsilon < b)$ $\quad = \begin{cases} 0 & x \leq a \\ 1 & a+\varepsilon \leq x \leq b-\varepsilon \\ 0 & x \geq b \end{cases}$

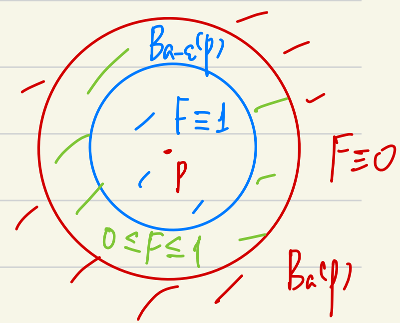


• Given $Q=[a_1,b_1]\times\cdots\times[a_n,b_n]\subset\mathbb{R}^n$.

define $F_Q(x_1,\cdots,x_n):=h_{a_1,b_1}(x_1)\cdots h_{a_n,b_n}(x_n)$



• Given $B_a(p)$, define $F_{B_a(p)}(\vec{x}):=g_{0,0+\varepsilon}(a-|\vec{x}-p|)$

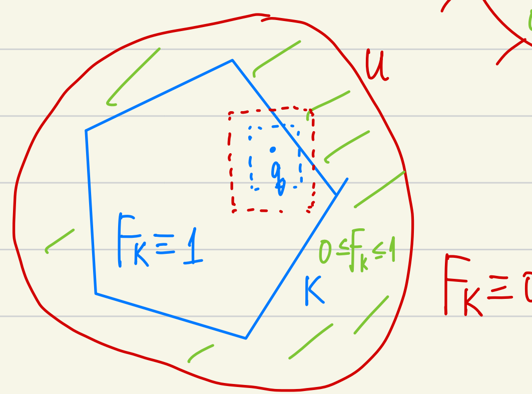


Thm 1: K compact $\subset$ U open set

$\exists$ bump function $F_K\in C^\infty$

s.t. $0\le F_K\le 1$ $\quad F_K\equiv 1$ on K

$F_K\equiv 0$ outside U



Pf: For each $q \in K$, define $F_q(x) = F_{Q_q} = \begin{cases} 1 & x \in \tilde{Q}_q \\ 0 & x \notin Q_q \end{cases}$ $\quad \tilde{Q}_q \subset Q_q \subset U$

K compact $\Rightarrow$ $\exists$ finite subcover $\{\tilde{Q}_{q_1}, \ldots, \tilde{Q}_{q_N}\}$

Let $F = \sum_{i=1}^{N} F_{q_i}$: smooth, $F \geq 1$ on K, $F = 0$ outside U.

Finally define $F_K = g_{0,1} \circ F$ $\quad$ · $x \in K \Rightarrow F \geq 1 \Rightarrow F_K = 1$

· $x \notin U \Rightarrow F = 0 \Rightarrow F_K = 0$ $\quad \square$

Lemma: Given $\mathcal{U} = \{U_\alpha\}_\alpha$ open cover of Ω, $\exists$ a sequence of (rectangle/ball) Q_i s.t

(1) $\{$interior of $Q_i\}$ is open cover of Ω

(2) $Q_i \subset U_\alpha$ for some α

(3) $\forall x \in \Omega$, $\exists$ open set intersecting Q_i for finitely many i.

Thm 2: Given open cover $\mathcal{U} = \{U_\alpha\}_\alpha$ of Ω, $\exists$ a partition of unity subordinate to $\mathcal{U}$

namely, a collection of smooth function $\{\rho_i\}$ s.t.

(1) $0 \le \rho_i \le 1$

(2) $\text{supp}(\rho_i) \subset U_\alpha$ for some α and compact

(3) $\forall x \in \Omega$ has an open set intersecting $\text{supp}(\rho_i)$ for finitely many i.

(4) $\sum_i \rho_i(x) = 1 \quad \forall x \in \Omega$.

Proof: Let F_{Q_i} bump function in Thm 1: $F_{Q_i} \equiv 1$ on Q_i, $F_{Q_i} \equiv 0$ outside U_α.

Define $\rho_i = F_{Q_i} \Big/ \sum_i F_{Q_i}$. Well-defined by (3)

$\square$

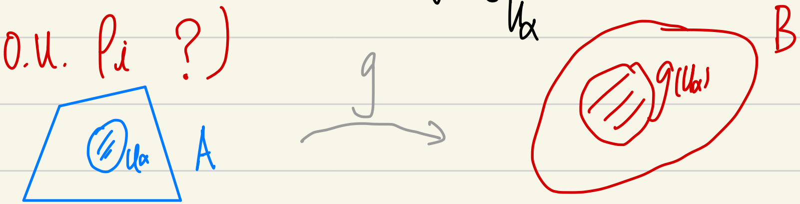
Applications:

(1) For $\{\rho_i\}$, $f = \sum_i \rho_i f$, and $\mathrm{supp}(\rho_i f) \subset U_\alpha$

Ω unbounded. Improper integral $\int_\Omega f\,dV := \sum_i \int_{U_\alpha} \rho_i \cdot f\,dV$

(Convergence, different P.O.U. ρ_i ?)

(2) Change of Variable Thm:



$\{U_\alpha\}$ open cover of A ; $\{\rho_i \circ g\}$ POU

$\{g(U_\alpha)\}$ open cover of B ; $\{\rho_i\}$ POU

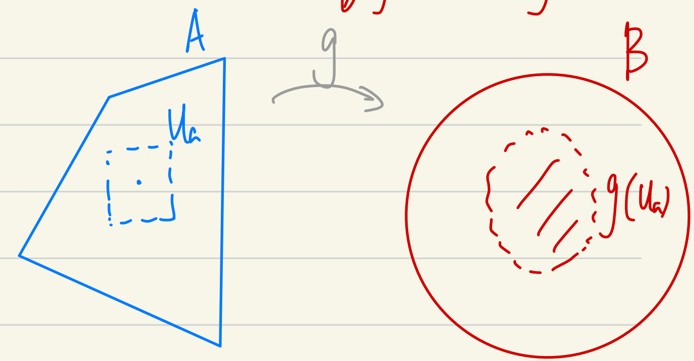
$$\int_A f\circ g\,|\det Dg|\,dV = \sum_i \int_{U_\alpha} (\rho_i \circ g)\cdot(f\circ g)\cdot|\det Dg|\,dV$$

$$\int_B f\,dV = \sum_i \int_{g(U_\alpha)} \rho_i \cdot f\,dV$$

Suffices to prove identity locally!

may assume $Dg(a)=id$

Proof of Change of Variables



- By the previous reduction, it suffices to prove

$\forall a \in A$, $\exists$ rectangle $U_a \subset A$ s.t.

$$\int_{Q_a} \cancelto{1}{f\circ g}\, |\det Dg|\, dV = \int_{g(Q_a)} \cancelto{1}{f}\, dV$$

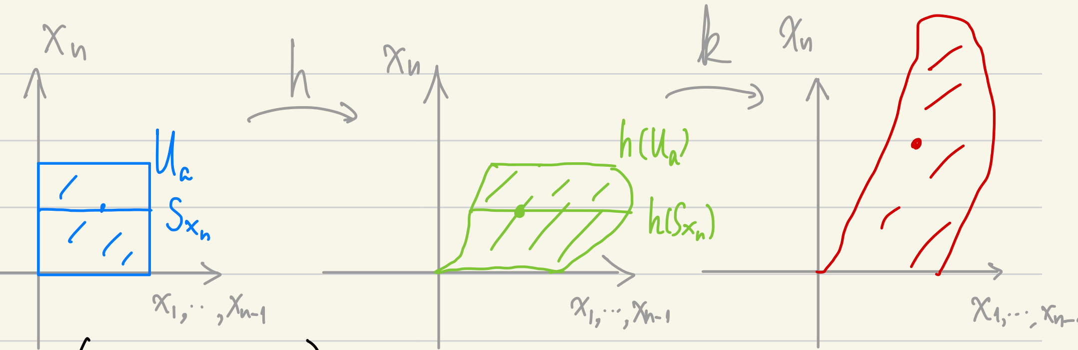
$D(T^{-1}\circ g)(a) = id$.

Furthermore, may assume $Dg(a)=id$. (if not, write $g = T\circ(T^{-1}\circ g)$)

- Locally, we can write $g = (y_1 = g_1(x_1,\cdots,x_n), \cdots, y_n = g_n(x_1,\cdots,x_n))$ as $k\circ h$,

where $h(x_1,\cdots,x_n) = (g_1,\cdots,g_{n-1},x_n)$; $k(x_1,\cdots,x_n) = (x_1,\cdots,x_{n-1},g_n(h^{-1}(x)))$

If $Dg = \left(\frac{\partial g_i}{\partial x_j}\right) = \left(\begin{array}{c|c} M & \\ \hline & \frac{\partial g_n}{\partial x_n} \end{array}\right)$



then $Dh = \left(\begin{array}{c|c} M & * \\ \hline 0 & 1 \end{array}\right)$ $\quad Dk = \left(\begin{array}{c|c} Id & 0 \\ \hline * & \frac{\partial g_n}{\partial x_n} \end{array}\right)$

- As $Dg(a) = id$ $\Rightarrow$ $\exists$ small nghd U of a s.t. $\boxed{\det M \neq 0}$ on U $\Rightarrow$ Dh, Dk invertible
- Inverse Function Thm $\Rightarrow$ local diffeomorphism i.e., $\exists\ U_a \subset U$ s.t. $\boxed{U_a \cong h(U_a) \cong k \circ h(U_a)}$

- Enough to prove the thm for h and k on U_a.

$$\int_{h(U_a)} 1 \cdot dV \underset{\text{Fubini}}{=} \int_{\min(x_n)}^{\max(x_n)} \int_{h(S_{x_n})} 1 \, dx_1 \cdots dx_{n-1} \, dx_n \underset{\text{Induction}}{=} \int_{\min(x_n)}^{\max(x_n)} \int_{S_{x_n}} |\det Dh| \, dx_1 \cdots dx_{n-1} \, dx_n \underset{\text{Fubini}}{=} \int_{U_a} |\det Dh| \, dV$$

Apply Change of Variable: $h\big|_{S_{x_n}}(x_1, \cdots, x_{n-1}) = (g_1, \cdots, g_{n-1})$

$$\det\left(Dh\big|_{S_{x_n}}\right) = \det M = \det Dh$$

$\square$